# Invitational Round

Junior Mathematicians' Problem Solving Competition

July 11th, 2021

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1. This is a fifteen question free-response test. Each question has exactly one integer answer.
2. You have 80 minutes to complete the test.
3. You will receive 15 points for each correct answer, and 0 points for each problem left unanswered or incorrect.
4. Figures are not necessarily drawn to scale.
5. No aids are permitted other than scratch paper, graph paper, rulers, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When you finish the exam, please stay in the Zoom meeting for further instructions.

# 1 Problem 1

The equation $ax^2 + 5x = 4$, where $a$ is some constant, has $x = 1$ as a solution. What is the other solution?

# 2 Problem 2

Two quadrilaterals are drawn on the plane such that they share no sides. What is the maximum possible number of intersections of the boundaries of the two quadrilaterals?

# 3 Problem 3

There are exactly 5 even positive integers less than or equal to 100 that are divisible by $x$. What is the sum of all positive integer values of $x$?

# 4 Problem 4

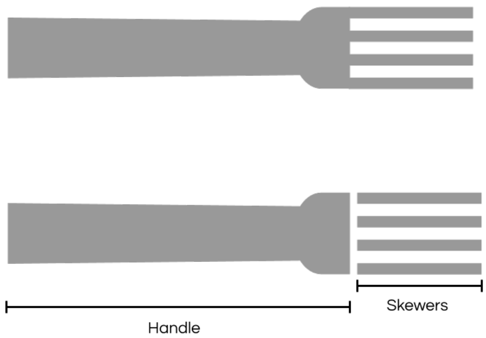
Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be sequences of real numbers such that $x_0 = 3$, $y_0 = 1$, and, for all positive integers $n$,

$$x_{n+1} + y_{n+1} = 2x_n + 2y_n,$$
$$x_{n+1} - y_{n+1} = 3x_n - 3y_n.$$

Find $x_5$.

# 5 Problem 5

An $n$-pointed fork is a figure that consists of two parts: a handle that weighs 12 ounces and $n$ ”skewers” that each weigh a nonzero integer weight (in ounces). Suppose $n$ is a positive integer such that there exists a fork with weight $n^2$. What is the sum of all possible values of $n$?




# 6 Problem 6

Five friends decide to meet together for a party. However, they did not plan the party well, and at noon, every friend leaves their own house and travels to one of the other four friends' houses, chosen uniformly at random. The probability that every friend sees another friend in the house they chose can be expressed in the form $\frac{m}{n}$. If $m$ and $n$ are relatively prime positive integers, find $m+n$.

# 7 Problem 7

In a $3 \times 3$ grid with nine square cells, how many ways can Jacob shade in some nonzero number of cells such that each row, column, and diagonal contains at most one shaded cell? (A diagonal is a set of squares such that their centers lie on a line that makes a $45°$ angle with the sides of the grid. Note that there are more than two diagonals.)

## 8 Problem 8

Let $x$ and $y$ be real numbers that satisfy

$$(x+y)^2(20x+21y)=12$$
$$(x+y)(20x+21y)^2=18.$$

Find $21x+20y$.

## 9 Problem 9

In $\triangle ABC$, let $D$ be on $\overline{AB}$ such that $AD=DC$. If $\angle ADC=2\angle ABC$, $AD=13$, and $BC=10$, find $AC$.

## 10 Problem 10

A point $P$ is chosen in isosceles trapezoid $ABCD$ with $AB=4$, $BC=20$, $CD=28$, and $DA=20$. If the sum of the areas of $PBC$ and $PDA$ is 144, then the area of $PAB$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime. Find $m+n$.

## 11 Problem 11

For some $n$, the arithmetic progression

$$4,9,14,\dots,n$$

has exactly 36 perfect squares. Find the maximum possible value of $n$.

## 12 Problem 12

Rectangle $ABCD$ is drawn such that $AB = 7$ and $BC = 4$. $BDEF$ is a square that contains vertex $C$ in its interior. Find $CE^2 + CF^2$.

## 13 Problem 13

Let $p$ be a prime and $n$ be an odd integer (not necessarily positive) such that

$$\frac{p^{n+p+2021}}{(p+n)^2}$$

is an integer. Find the sum of all distinct possible values of $p \cdot n$.

## 14 Problem 14

Let there be a $\triangle ACD$ such that $AC = 5$, $AD = 12$, and $CD = 13$, and let $B$ be a point on $AD$ such that $BD = 7$. Let the circumcircle of $\triangle ABC$ intersect hypotenuse $CD$ at $E$ and $C$. Let $AE$ intersect $BC$ at $F$. If the ratio $\frac{FC}{BF}$ can be expressed as $\frac{m}{n}$ where $m$ and $n$ are relatively prime, find $m + n$.

## 15 Problem 15

Abhishek is choosing positive integer factors of $2021 \times 2^{2021}$ with replacement. After a minute passes, he chooses a random factor and writes it down. Abhishek repeats this process until the first time the product of all numbers written down is a perfect square. Find the expected number of minutes it takes for him to stop.

[1] [2]

[1] The publication, reproduction or communication of the problems or solutions of all JMPSC exams during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

[2] The team on the Junior Mathematician's Problem Solving Contest (JMPSC) reserves the right to re-examine students before deciding whether to grant official status to their scores.